the branch-line couplers and the sum-and-difference networks are of appreciable magnitude for only three cycles or less of the carrier frequency. Thus, in passing through these components, pulses only three cycles long, or spaced from each other by only three cycles, retain their general shape and identity.

Conclusion

It has been shown that the pulse responses of microwave components, made of nondispersive transmission lines only, are sums of replicas of the applied pulse. Two different ways were described by which the amplitudes and times of occurrence of the individual replicas can be found from the component frequency responses or impulse responses.

This technique for finding pulse responses was applied to stepped transmission-line transformers, to the backward coupler as a hybrid and sum-and-difference networks, and to branch-line couplers as hybrids and sum-and-difference networks. It was found that rectangular-pulse envelopes lasting for only three periods of the carrier frequency would pass through any one of these components without extreme distortion.

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Sets of Eigenvectors for Volumes of Revolution*

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Summary—**The electric and magnetic eigenvectors of a volume of revolution can be written in terms of two-dimensional scalar and vector functions. These functions are the eigenfunctions of certain linear transformations in the meridian plane. The form of the transformation is examined, and much attention is devoted to the orthogonality properties of their eigenfunctions and the calculation of their eigenvalues from variational principles.**

Among the sets of eigenvectors which exist in a finite three-dimensional volume, the "electric" and "magnetic" modes are of particular importance for the calculation of electric and magnetic fields. The purpose of the present paper is to investigate the properties of these modes in volumes of revolution of the kind depicted in Fig. 1. An explicit mathematical expression can be given for the modes of a few simple volumes, such as the sphere and the coaxial cylinder, but in the most general case one has to resort to approximate procedures to obtain quantitative data. The most frequently used methods rely on the replacement of differential equations by difference equations, and on the use of variational principles for the calculation of eigenvalues. It is necessary, for a systematic application of these methods, to possess a precise classification and enumeration of the modes and their characteristics. This is what this paper, inspired by a previous analysis by Bernier,[1] sets out to provide.

The first structure to be examined will be the toroidal volume of Fig. 1(a), which is of importance for circular particle accelerators and, more generally, for ring-like structures through which particles or fluids are flowing. The fact that a toroidal volume does not contain any portion of the axis of revolution facilitates the mathematical formulation of the problem.

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[1] J. Bernier, "On electromagnetic resonators," *Onde élect.*, vol. 26, pp. 305–317; August–September, 1946.

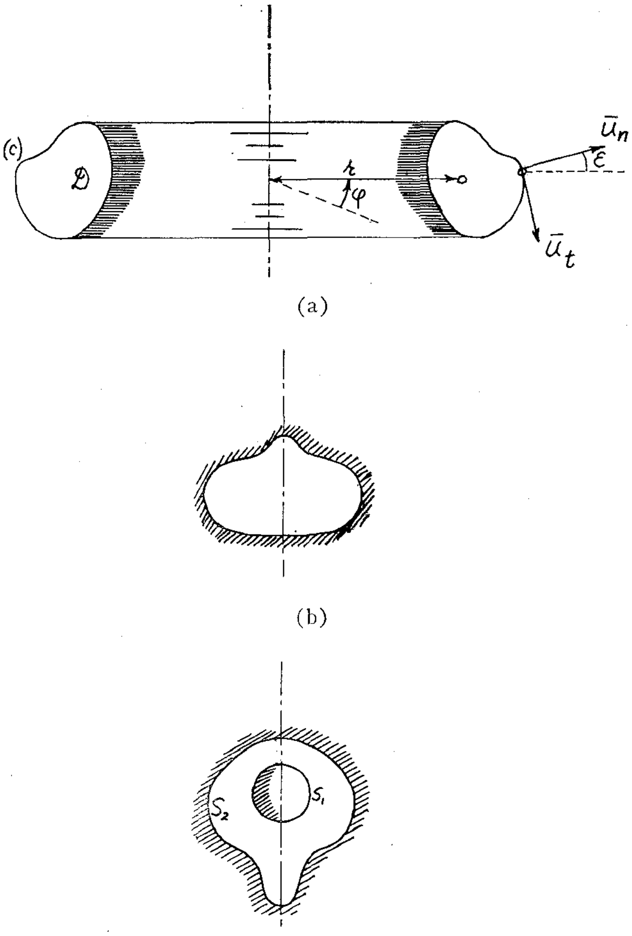

Fig. 1—Cavities of revolution.

I. Preliminary Remarks

A. Fields in Volumes of Revolution

One of the problems to be investigated is the determination of the expansion coefficients of a piecewise continuous vector function $\bar{a}(r, z, \phi)$ (r, z, and ϕ are cylindrical coordinates). This determination is simplified by a preliminary Fourier expansion in ϕ.

$$\bar{a}(r, z, \phi) = \bar{p}_0(r, z) + v_0(r, z)\bar{u}_\phi + \sum_m [\sin m\phi \cdot \bar{p}_m(r, z) + \cos m\phi \cdot \bar{q}_m(r, z)] + \sum_m [(-w_m(r, z) \sin m\phi + v_m(r, z) \cos m\phi) \cdot \bar{u}_\phi]. \quad (1)$$

The $\bar{p}$'s and $\bar{q}$'s are "meridian" vectors (*i.e.*, vectors situated in the meridian plane). Vectors such as $v_0\bar{u}_\phi$, where $\bar{u}_\phi$ is a unit vector perpendicular to the meridian plane and directed toward increasing ϕ, form the "circular" components. The divergence and curl of $\bar{a}$ are given by

$$\operatorname{div} \bar{a} = \operatorname{div}_M \bar{p}_0 + \sum_m \sin m\phi \cdot \left(\operatorname{div}_M \bar{p}_m - \frac{mv_m}{r}\right) + \sum_m \cos m\phi \cdot \left(\operatorname{div}_M \bar{q}_m - \frac{mw_m}{r}\right), \quad (2)$$

$$\operatorname{curl} \bar{a} = \operatorname{curl}_M \bar{p}_0 + \operatorname{curl}(v_0\bar{u}_\phi) + \sum_m \sin m\phi \cdot \left[\operatorname{curl}_M \bar{p}_m - \operatorname{curl}_M (w_m\bar{u}_\phi) - \frac{m}{r}(\bar{u}_\phi \times \bar{q}_m)\right] + \sum_m \cos m\phi \cdot \left[\operatorname{curl}_M \bar{q}_m + \operatorname{curl}_M (v_m\bar{u}_\phi) + \frac{m}{r}(\bar{u}_\phi \times \bar{p}_m)\right]. \quad (3)$$

Differential operators having the subscript M (M for meridian) are obtained from the usual forms by dropping derivatives with respect to ϕ and (for meridian vectors) ϕ projections.

When $\bar{a}$ is solenoidal (*i.e.*, $\operatorname{div} \bar{a}=0$), the following relations hold:

$$\operatorname{div}_M \bar{p}_0 = 0, \quad \operatorname{div}_M \bar{p}_m = \frac{mv_m}{r}, \quad \operatorname{div}_M \bar{q}_m = \frac{mw_m}{r}. \quad (4)$$

When $\bar{a}$ is irrotational (*i.e.*, $\operatorname{curl} \bar{a}=0$),

$$\operatorname{curl}_M \bar{p}_0 = \operatorname{curl}_M \bar{p}_m = \operatorname{curl}_M \bar{q}_m = \operatorname{curl}_M (v_0\bar{u}_\phi) = 0,$$

$$\operatorname{curl}_M (v_m\bar{u}_\phi) = -\frac{m}{r}(\bar{u}_\phi \times \bar{p}_m),$$

$$\operatorname{curl}_M (w_m\bar{u}_\phi) = -\frac{m}{r}(\bar{u}_\phi \times \bar{q}_m). \quad (5)$$

B. Electric Eigenvectors

The electric eigenvectors of a simply-bounded volume fall into two categories:

1) Irrotational eigenvectors $\bar{f}_{mnp} = \operatorname{grad} \psi_{mnp}$ where ψ_{mnp} is an eigenfunction of

$$\nabla^2\psi_{mnp} + \lambda'_{mnp}\psi_{mnp} = 0$$
$$\psi_{mnp} = 0 \text{ on boundary surface } S. \quad (6)$$

The triple index accounts for the triple infinity of eigenfunctions.

2) Solenoidal eigenvectors $\bar{e}_{mnp}$, solutions of

$$-\operatorname{curl}\operatorname{curl} \bar{e}_{mnp} + \lambda''_{mnp}\bar{e}_{mnp} = 0$$
$$\bar{u}_n \times \bar{e}_{mnp} = 0 \text{ on boundary surface } S. \quad (7)$$

The notation $\bar{u}$ stands for "unit vector," and $\bar{u}_n$ is the unit vector along the outward-pointing normal to S.

C. Magnetic Eigenvectors

The complete set of magnetic eigenvectors of a toroidal volume consists of

1) A single "sourceless" vector $\bar{h}_0 = \text{grad}\,\theta_0$, tangent to the boundary surface.[2]
2) Irrotational eigenvectors $\bar{g}_{mnp} = \text{grad}\,\theta_{mnp}$, where θ_{mnp} is an eigenfunction of

$$\nabla^2 \theta_{mnp} + \nu_{mnp}\theta_{mnp} = 0 \qquad \frac{\partial \theta_{mnp}}{\partial n} = 0 \text{ on } S. \tag{8}$$

3) Solenoidal eigenvectors $\bar{h}_{mnp}$, solutions of

$$-\text{curl curl}\,\bar{h}_{mnp} + \mu_{mnp}\bar{h}_{mnp} = 0$$
$$\bar{u}_n \times \text{curl}\,\bar{h}_{mnp} = 0 \text{ on } S. \tag{9}$$

It can be shown that the eigenvalues μ and λ'' are identical, and that the electric and magnetic solenoidal eigenvectors are multiples of the curl of each other. In other words, $\bar{e}_{mnp}$ is proportional to curl $\bar{h}_{mnp}$ and $\bar{h}_{mnp}$ is proportional to curl $\bar{e}_{mnp}$. The proportionality constants depend on the normalization of the eigenvectors.

D. Variational Principle for Eigenvalues

Variational properties are of considerable interest for the approximate determination of eigenvalues and eigenvectors when the boundaries are irregular in shape. The basic property is as follows: when $\mathfrak{L}$ is a negative definite self-adjoint linear transformation,[3] all eigenvalues in $\mathfrak{L}u_n + \lambda_n u_n = 0$ are real and positive. Denoting by $\langle a, b\rangle$ the scalar product of a by b, the lowest eigenvalue λ_1 is the minimum of

$$J(u) = -\frac{\langle \mathfrak{L}u, u\rangle}{\langle u, u\rangle}.$$

This minimum is attained for the lowest eigenfunction u_1. The functions admitted for competition (the "admissible" functions) must belong to the space of definition of the transformation $\mathfrak{L}$. The second lowest eigenvalue is the minimum value of J with respect to admissible functions that are orthogonal to u_1 (*i.e.*, for which $\langle u, u_1\rangle = 0$), and the minimum is attained for $u = u_2$. Similarly, λ_n is the minimum of J with respect to u's that are orthogonal to the $(n-1)$ lowest eigenfunctions, and the minimum is attained for $u = u_n$. Similar results are obtained, *mutatis mutandis*, for positive-definite transformations.

These considerations can be applied to transformations (6) and (8). The scalar product to be used here is $\iiint_V ab\,dV$, and the λ'_{mnp} are obtained as stationary values of

$$J(\psi) = -\frac{\iiint_V \psi\nabla^2\psi\,dV}{\iiint_V \psi^2\,dV}. \tag{10}$$

The admissible functions vanish on the boundary, and are continuous up to their second derivatives. The eigenvalues ν_{mnp} are obtained as stationary values of the same expression, the admissible functions having the same continuity properties, but a vanishing normal derivative on S.

Transformation (7) with scalar product $\langle \bar{a}, \bar{b}\rangle = \iiint_V \bar{a}\cdot\bar{b}\,dV$ leads to the characterization of λ''_{mnp} as stationary value of

$$J(\bar{e}) = \frac{\iiint_V \bar{e}\cdot\text{curl curl}\,\bar{e}\,dV}{\iiint_V \bar{e}\cdot\bar{e}\,dV} \tag{11}$$

where the admissible vectors have zero divergence, are continuous up to their second derivatives, and are perpendicular to the boundary surface.

II. Electric Modes in Toroidal Volumes of Revolution

The general considerations of the preceding paragraph will now be applied more specifically to volumes of revolution.

A. Irrotational Eigenvectors

The general expression for these eigenvectors is

$$\bar{f}_{mnp} = \text{grad}[\sin m\phi\cdot\alpha_{mnp}(r, z)]$$
$$= \sin m\phi\cdot\text{grad}_M\,\alpha_{mnp} + \frac{m\cos m\phi}{r}\alpha_{mnp}\,\bar{u}_\phi. \tag{12}$$

The functions α are eigenfunctions of

$$\left(\nabla_M{}^2 - \frac{m^2}{r^2}\right)\alpha_{mnp} + \lambda'_{mnp}\,\alpha_{mnp} = 0$$

with

$$\alpha_{mnp} = 0 \text{ on } C. \tag{13}$$

Modes of revolution are obtained by setting $m = 0$ in (13). For ϕ-dependent modes, the usual ϕ degeneracy is encountered; *i.e.*, two modes, grad$[\sin m\phi\cdot\alpha]$ and grad$[\cos m\phi\cdot\alpha]$, correspond to each value of λ'. This characteristic property will be found for all other ϕ-dependent modes to be examined in the future. For reasons of conciseness, only one of the modes will be written down explicitly. The second one can then be obtained simply by increasing $m\phi$ by $\pi/2$.

[2] We define a "sourceless" vector as having zero divergence and zero curl.

[3] $\mathfrak{L}$ is self-adjoint when $\langle u, \mathfrak{L}v\rangle = \langle \mathfrak{L}u, v\rangle$ for all u, v belonging to the space of definition of $\mathfrak{L}$, and it is negative definite when $\langle u, \mathfrak{L}u\rangle \le 0$, the equality sign being obtained for, and only for, $u = 0$. These properties are associated with a specific definition of the scalar product $\langle a, b\rangle$.

The transformation associated with (13) is self-adjoint and negative definite with respect to the scalar product $\iint_D a \cdot b \cdot rdrdz$. The α_{mnp} are orthogonal in the sense that $\iint_D \alpha_{mnp}\, \alpha_{mn'p'} rdrdz = 0$ for $(n, p) \neq (n', p')$. The norms of $\bar{f}$ and α are related by

$$\iiint_V |\bar{f}_{mnp}|^2 \cdot dV = \pi \iint_D \left[(\text{grad } \alpha_{mnp})^2 + \frac{m^2}{r^2} \alpha_{mnp}^2 \right] rdrdz = \lambda'_{mnp} \cdot \pi \iint_D \alpha_{mnp}^2 rdrdz. \quad (14)$$

The eigenvalues λ'_{mnp} can be obtained as stationary values of

$$J(\alpha) = - \frac{\iint_D \alpha \left[\nabla_M^2 \alpha - \frac{m^2 \alpha}{r^2} \right] rdrdz}{\iint_D \alpha^2 rdrdz}. \quad (15)$$

The admissible functions vanish on boundary (c) and have continuous derivatives up to the second order.

B. Solenoidal Eigenvectors

1) Modes of Revolution: The solenoidal eigenvectors $\bar{e}_{onp}$ can usefully be split into a meridian and a circular part according to

$$\bar{e}_{onp} = \bar{c}_{onp}(r, z) + \beta_{onp}(r, z) \bar{u}_\phi.$$

If the latter expression is substituted into (7), and the ϕ independence taken into account, uncoupled equations are obtained for $\bar{c}$ and β. The modes are consequently of two different sorts.

a) Circular modes $\beta_{onp}\bar{u}_\phi$: There is a double infinity of these modes, corresponding to the eigenfunctions of

$$\nabla_M^2 \beta_{onp} - \frac{\beta_{onp}}{r^2} + \lambda''_{onp} \beta_{onp} = 0 \quad \beta_{onp} = 0 \text{ on } C. \quad (16)$$

The β_{onp} are, in consequence, equal to the functions α_{1np} encountered in Section II, A, and partake of their orthogonality and stationarity properties. The eigenvalues λ''_{onp} of the circular modes are equal to the λ'_{1np}. The normalization is particularly simple:

$$\iiint_V \beta_{onp} \bar{u}_\phi \cdot \beta_{onp} \bar{u}_\phi dV = 2\pi \iint_D \beta_{onp}^2 rdrdz. \quad (17)$$

An example of application of variational principle (15) to calculate λ''_{onp} is given[4] in Fig. 2.

[4] For more details, see D. F. Meronek and J. Van Bladel, "Resonant modes and frequencies of a cigar-shaped cavity," *Microwave J.*, pp. 32–33; May, 1959.

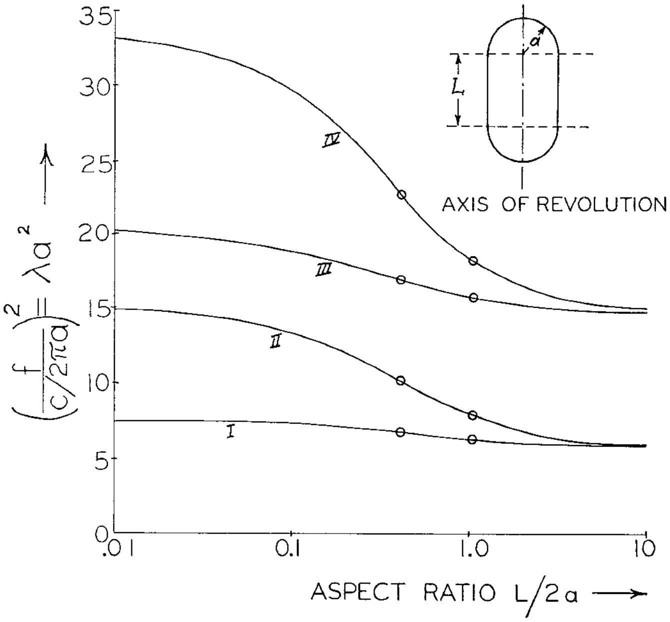

Fig. 2—Resonant frequencies of a cigar-shaped cavity. The resonant frequency is related to the eigenvalue λ^2 by $\lambda^2 = (2\pi f/c)^2$. Curves 1 and 2 correspond to the lowest two values of λ'''_{onp} (20). Curves 3 and 4 correspond to the lowest two values of λ''_{onp} (16). The circles represent experimentally determined points. (Reproduced with permission of the *Microwave Journal.*)

b) Meridian modes $\bar{c}_{onp}$: The eigenvalue problem satisfied by the $\bar{c}_{onp}$ is

$$-\text{curl curl } \bar{c}_{onp} + \lambda'''_{onp} \bar{c}_{onp} = 0$$

$$\text{with } \begin{cases} \bar{u}_n \times \bar{c}_{onp} = 0 \\ \text{div}_M \bar{c}_{onp} = 0 \end{cases} \text{ on } (C). \quad (18)$$

It is a simple matter to show that these meridian vectors are actually the curl of the circular magnetic eigenvectors. More precisely, the $\bar{c}_{onp}$ can be put in the form,

$$\bar{c}_{onp} = \text{curl } [\delta_{onp} \bar{u}_\phi] = - \frac{\partial \delta_{onp}}{\partial z} \bar{u}_r + \left[\frac{\partial \delta_{onp}}{\partial r} + \frac{\delta_{onp}}{r} \right] \bar{u}_z = \frac{1}{r} [\text{grad } (\delta_{onp} r) \times \bar{u}_\phi], \quad (19)$$

where the functions δ_{onp} satisfy the eigenvalue problem,

$$\left(\nabla_M^2 - \frac{1}{r^2} \right) \delta_{onp} + \lambda'''_{onp} \delta_{onp} = 0$$

with

$$\bar{u}_n \times \text{curl } [\delta_{onp} \bar{u}_\phi] = 0 \text{ on } (C). \quad (20)$$

The boundary condition can be rewritten in the form,

$$\frac{1}{r} \frac{\partial}{\partial n} (\delta_{onp} r) = \frac{\partial \delta_{onp}}{\partial n} + \frac{\delta_{onp}}{r} \cdot \cos \epsilon = 0. \quad (21)$$

The δ_{onp} are orthogonal with respect to a scalar product $\int\int_D abrdrdz$, and the eigenvalues are the stationary values of (15) with $m=1$. The admissible functions are required to possess the usual continuity properties, and to satisfy (21) at the boundary. The normalization integral is simply

$$\int\int\int_v \bar{c}_{onp}\cdot\bar{c}_{onp}dV = 2\pi\cdot\lambda'''_{onp}\int\int_D \delta^2_{onp}r\,drdz. \quad (22)$$

2) Azimuth-Dependent Modes: The periodicity of the modes indicates that the general form of $\bar{e}_{mnp}$ is

$$\bar{e}_{mnp} = \bar{c}_{mnp}\sin m\phi + \bar{c}'_{mnp}\cos m\phi + [\beta_{mnp}\cos m\phi - \beta'_{mnp}\sin m\phi]\bar{u}_\phi.$$

If the latter expression is inserted in (7), uncoupled and identical equations are obtained for the pairs ($\bar{c}$, β) and ($\bar{c}'$, β'). This fact indicates the existence of an eigenvector $\bar{c}_{mnp}\sin m\phi+\beta_{mnp}\cos m\phi\bar{u}_\phi$, and also of an eigenvector ($\bar{c}'$, $-\beta'$) obtained from the former by increasing $m\phi$ by $\pi/2$, *i.e.*, by rotating the configuration through an angle $\pi/2m$. The equations which $\bar{c}_{mnp}$ and β_{mnp} are required to satisfy are rather complicated. Dropping the subscripts for a moment, they turn out to be

$$-\mathrm{curl}_M\,\mathrm{curl}_M\,\bar{c} - \frac{m^2\bar{c}}{r^2} + \frac{m}{r}\cdot\mathrm{grad}\,\beta + \frac{m\beta}{r^2}\bar{u}_r + \lambda''\bar{c} = 0; \quad (23)$$

$$\nabla^2_M\beta - \frac{\beta}{r^2} + \frac{2m}{r^2}(\bar{c}\cdot\bar{u}_r) - \frac{m}{r}\mathrm{div}_M\,\bar{c} + \lambda''\beta = 0. \quad (24)$$

These equations can be simplified by taking into account the fact that $\bar{e}_{mnp}$ is solenoidal; *i.e.*, that

$$\mathrm{div}\,[\bar{c}\sin m\phi + \beta\cos m\phi\bar{u}_\phi] = \sin m\phi\left[\mathrm{div}_M\,\bar{c} - \frac{m\beta}{r}\right] = 0.$$

There exists, in consequence, a relation between β and $\bar{c}$, namely,

$$\beta = \frac{r}{m}\mathrm{div}_M\,\bar{c}. \quad (25)$$

Upon substitution of this expression in (23), an equation for $\bar{c}$ alone is obtained.

$$\nabla^2_M\bar{c}_{mnp} - \frac{m^2}{r^2}\bar{c}_{mnp} + \bar{u}_r\frac{2}{r}\mathrm{div}_M\,\bar{c}_{mnp} + \lambda''_{mnp}\bar{c}_{mnp} = 0$$

$$\text{with}\begin{cases}\bar{u}_n\times\bar{c}_{mnp} = 0\\ \mathrm{div}_M\,\bar{c}_{mnp} = 0\end{cases}\text{on }(C). \quad (26)$$

The meridian part of a solenoidal eigenvector must, in consequence, be an eigenvector of (26). Conversely, to each eigenvector of (26) corresponds an eigenvector,

$$\bar{c}_{mnp}\cdot\sin m\phi + \frac{r}{m}\mathrm{div}_M\,\bar{c}\cdot\cos m\phi\bar{u}_\phi, \quad (27)$$

of the original three-dimensional problem (7). It is important to list orthogonality and stationarity properties of the $\bar{c}_{mnp}$. These properties can be obtained from the general equation (9) wherein (27) is substituted. They can also be established directly from a study of the transformation,

$$\mathfrak{L}\bar{v} = \nabla^2_M\bar{v} - \frac{m^2}{r^2}\bar{v} + \frac{2\bar{u}_r}{r}\mathrm{div}_M\,\bar{v}$$

$$\text{with}\begin{cases}u_n\times\bar{v} = 0\\ \mathrm{div}_M\bar{v} = 0\end{cases}\text{on } C, \quad (28)$$

in the meridian plane. The relevant steps are collected in the Appendix. It turns out that the scalar product which is suited to the problem is

$$\langle\bar{v},\bar{w}\rangle = \int\int_D\left[\bar{v}\cdot\bar{w} + \frac{r^2}{m^2}\mathrm{div}_M\,\bar{v}\cdot\mathrm{div}_M\,\bar{w}\right]rdrdz \quad (29)$$

where $\bar{v}$ and $\bar{w}$ are two meridian vectors. With the latter definition of the scalar product, transformation $\mathfrak{L}$ is self-adjoint and negative-definite, the eigenvectors are orthogonal in the sense that

$$\langle\bar{c}_{mnp},\bar{c}_{mn'p'}\rangle = \int\int_D\left[\bar{c}_{mnp}\cdot\bar{c}_{mn'p'} + \frac{r^2}{m^2}\mathrm{div}_M\,\bar{c}_{mnp}\cdot\mathrm{div}_M\,\bar{c}_{mn'p'}\right]rdrdz = 0$$

$$\text{for } (n, p) \neq (n'p')$$

and the eigenvalues λ''_{mnp} are obtained from the stationarity properties of

$$J(\bar{c}) = -\frac{\langle\mathfrak{L}\bar{c},\bar{c}\rangle}{\langle\bar{c},\bar{c}\rangle} = -\frac{\int\int_D\left[\bar{c}\cdot\mathfrak{L}\bar{c} + \frac{r^2}{m^2}\mathrm{div}_M\,\bar{c}\cdot\mathrm{div}_M\,\mathfrak{L}\bar{c}\right]rdrdz}{\int\int_D\left[\bar{c}\cdot\bar{c} + \frac{r^2}{m^2}\mathrm{div}_M\,\bar{c}\cdot\mathrm{div}_M\,\bar{c}\right]rdrdz} \quad (30)$$

where the $\bar{c}$ have continuous derivatives up to the second, and satisfy the boundary conditions evidenced in (28). Third order derivatives appear in the numerator, An equivalent expression for the latter can be derived which involves lesser order derivatives only. The derivation is based on

$$\iint_D \left\{ \bar{c}\cdot\mathfrak{L}\bar{c} + \frac{r^2}{m^2}\operatorname{div}_M \bar{c}\operatorname{div}_M \mathfrak{L}\bar{c} + \left|\operatorname{curl}_M \bar{c}\right|^2 + \left[\frac{m}{r}\bar{u}_\phi \times \bar{c} + \operatorname{curl}_M\left(\frac{r}{m}\operatorname{div}_M \bar{c}\right)\bar{u}_\phi\right]^2\right\} rdrdz$$

$$= \int_c \left\{(\bar{u}_n \times \bar{c})\cdot\operatorname{curl}_M \bar{c} - \operatorname{div}_M \bar{c}(\bar{c}\cdot\bar{u}_n) - \frac{r}{m}\operatorname{div}_M \bar{c}\left[\bar{u}_t\cdot\operatorname{curl}_M\left(\frac{r}{m}\operatorname{div}_M \bar{c}\right)\bar{u}_\phi\right]\right\} rdc, \quad (31)$$

a direct consequence of the substitution of (27) in the general relation,

$$\iiint_v [-\bar{v}\cdot\operatorname{curl}\operatorname{curl}\bar{v} + \operatorname{curl}\bar{v}\cdot\operatorname{curl}\bar{v}]\, dv = \iint_S (\bar{v}\times\operatorname{curl}\bar{v})\cdot\bar{u}_n dS. \quad (32)$$

The right-hand member of (31) vanishes for all admissible vectors. As a consequence, $J(\bar{c})$ can be rewritten as $J(\bar{c}) = N/D$ with

$$N = \iint_D \left\{\left[\frac{mc_z}{r} - \frac{r}{m}\frac{\partial^2 c_r}{\partial r\partial z} - \frac{1}{m}\frac{\partial c_r}{\partial z} - \frac{r}{m}\frac{\partial^2 c_z}{\partial z^2}\right]^2 + \left[\frac{\partial c_r}{\partial z} - \frac{\partial c_z}{\partial r}\right]^2 + \left[\frac{r}{m}\frac{\partial^2 c_r}{\partial r^2} + \frac{3}{m}\cdot\frac{\partial c_r}{\partial r} + \frac{2}{m}\cdot\frac{\partial c_z}{\partial z} + \frac{r}{m}\cdot\frac{\partial^2 c_z}{\partial r\partial z} + \frac{c_r}{r}\left(\frac{1}{m} - m\right)\right]^2\right\} rdrdz$$

$$D = \iint_D \left[c_r^2 + c_z^2 + \frac{r^2}{m^2}\left(\frac{\partial c_r}{\partial r} + \frac{c_r}{r} + \frac{\partial c_z}{\partial z}\right)^2\right] rdrdz.$$

This form is suitable for numerical computations. We repeat that the admissible vectors must satisfy the conditions,

$$\operatorname{div}_M \bar{c} = \frac{\partial c_r}{\partial r} + \frac{c_r}{r} + \frac{\partial c_z}{\partial z} = 0,$$

$$|\bar{c}\times\bar{u}_n| = c_r \sin\epsilon - c_z\cos\epsilon = 0, \quad (33)$$

at the boundary.

Finally, the normalization relations are

$$\iiint_V \bar{c}_{mnp}\cdot\bar{c}_{mnp}\, dV = \pi\iint_D \left[\bar{c}_{mnp}\cdot\bar{c}_{mnp} + \frac{r^2}{m^2}(\operatorname{div}_M \bar{c}_{mnp})^2\right] rdrdz. \quad (34)$$

III. Expansion in Electric Eigenvectors

We now turn to the task of determining the coefficients of expansion of the vector function $\bar{a}$ considered in the first paragraph.[5] The expansion breaks down in separate expansions involving the various Fourier coefficients:

$$\bar{p}_0(r,z) = \sum_n\sum_p A_{onp}\operatorname{grad}\alpha_{onp} + \sum_n\sum_p D_{onp}\bar{c}_{onp},$$

$$v_0(r,z) = \sum_n\sum_p C_{onp}\beta_{onp},$$

$$\bar{p}_m(r,z) = \sum_n\sum_p A_{mnp}\operatorname{grad}\alpha_{mnp} + \sum_n\sum_p E_{mnp}\bar{c}_{mnp},$$

$$\bar{q}_m(r,z) = \sum_n\sum_p B_{mnp}\operatorname{grad}\alpha_{mnp} + \sum_n\sum_p F_{mnp}\bar{c}_{mnp},$$

$$v_m(r,z) = \sum_n\sum_p A_{mnp}\frac{m}{r}\alpha_{mnp} + \sum_n\sum_p E_{mnp}\frac{r}{m}\operatorname{div}_M \bar{c}_{mnp},$$

$$w_m(r,z) = \sum_n\sum_p B_{mnp}\frac{m}{r}\alpha_{mnp} + \sum_n\sum_p F_{mnp}\frac{r}{m}\operatorname{div}_M \bar{c}_{mnp}. \quad (35)$$

The value of the coefficients can be calculated from (1) and (35). Results only will be quoted. For the irrotational terms:

$$A_{onp} = \frac{\iint_D \bar{p}_o\cdot\operatorname{grad}\alpha_{onp} rdrdz}{\lambda'_{onp}\iint_D \alpha_{onp}^2 rdrdz} = -\frac{\iint_D \alpha_{onp}\operatorname{div}_M \bar{p}_o rdrdz}{\lambda'_{onp}\iint_D \alpha_{onp}^2 rdrdz}, \quad (36)$$

$$A_{mnp} = \frac{\iint_D \left[\bar{p}_m\cdot\operatorname{grad}\alpha_{mnp} + v_m\cdot\frac{m}{r}\cdot\alpha_{mnp}\right] rdrdz}{\lambda'_{mnp}\iint_D \alpha_{mnp}^2 rdrdz} = -\frac{\iint_D \left[\operatorname{div}_M \bar{p}_m - \frac{m}{r}v_m\right]\alpha_{mnp} rdrdz}{\lambda'_{mnp}\iint_D \alpha_{mnp}^2 rdrdz}. \quad (37)$$

A similar expression can be obtained for B_{mnp} by substituting $\bar{q}_m$ and w_m for $\bar{p}_m$ and v_m, respectively. Formulas (2), (36), and (37) indicate that coefficients A and B vanish when $\bar{a}$ is solenoidal.

[5] Simpler formulas are obtained when $\bar{a}$ is specialized to be an electric or a magnetic field. This specialization will be considered in a subsequent paper where the application to particle accelerator problems will be emphasized.

For the solenoidal terms:

$$C_{onp} = \frac{\iint_D v_o \cdot \beta_{onp} r dr dz}{\iint_D \beta_{onp}^2 r dr dz} = \frac{\iint_D \text{curl}_M (\beta_{onp}\bar{u}_\phi)\cdot \text{curl}(v_o \bar{u}_\phi) r dr dz - \int_c v_o[\bar{u}_t \cdot \text{curl}(\beta_{onp}\bar{u}_\phi)] r dc}{\lambda'''_{onp} \iint_D \beta_{onp}^2 r dr dz},$$

$$D_{onp} = \frac{\iint_D \bar{p}_o \cdot \bar{c}_{onp} r dr dz}{\iint_D |\bar{c}_{onp}|^2 r dr dz} = \frac{\iint_D \text{curl}_M \bar{p}_o \cdot \text{curl}_M \bar{c}_{onp} r dr dz - \int_c (\bar{u}_n \times \bar{p}_o)\cdot \text{curl}_M \bar{c}_{onp} r dc}{\lambda''_{onp} \iint_D |\bar{c}_{onp}|^2 r dr dz}$$

$$= \frac{\iint_D \delta_{onp}(\text{curl}\, \bar{p}_o \cdot \bar{u}_\phi) r dr dz - \int_c \delta_{onp}(\bar{p}_0 \cdot \bar{u}_t) r dc}{\iint_D \delta_{onp}^2 r dr dz},$$

$$E_{mnp} = \frac{\iint_D \left[\bar{p}_m \cdot \bar{c}_{mnp} + \frac{r}{m} v_m \,\text{div}_M \bar{c}_{mnp}\right] r dr dz \cdot}{\iint_D \left[|\bar{c}_{mnp}|^2 + \frac{r^2}{m^2}(\text{div}_M \bar{c}_{mnp})^2\right] r dr dz}. \tag{38}$$

The numerator can be rewritten as

$$\frac{1}{\lambda''_{mnp}} \iint_D \Bigg\{ \text{curl}_M \bar{p}_m \cdot \text{curl}_M \bar{c}_{mnp} + \left[\text{curl}(v_m \bar{u}_\phi) + \frac{m}{r}(\bar{u}_\phi \times \bar{p}_m)\right] \cdot \left[\text{curl}\left(\bar{u}_\phi \frac{r}{m} \text{div}_M \bar{c}_{mnp}\right) + \frac{m}{r}(\bar{u}_\phi \times \bar{c}_{mnp})\right] r dr dz.$$

A similar expression can be obtained for F_{mnp} by substituting $\bar{q}_m$ and w_m for $\bar{p}_m$ and v_m, respectively. It will be noticed that the $C, D, E,$ and F vanish when $\bar{a}$ is irrotational [which, according to (3), entails vanishing of the surface integral in the numerator], and perpendicular to the boundary (which entails $\bar{u}_n \times \bar{p}_m = 0$ and $v_m = 0$, *i.e.*, vanishing of the line integral in the numerator).

IV. Magnetic Modes in Toroidal Volumes of Revolution

The complete set of eigenvectors includes, first of all, a "sourceless" vector $\bar{u}_\phi / r$. It includes, in addition, a triple infinity of irrotational eigenvectors and a triple infinity of solenoidal eigenvectors. These we now proceed to investigate.

A. Irrotational Eigenvectors

The irrotational eigenvectors are of the form $\bar{g}_{mnp} = \text{grad}\,[\sin m\phi \cdot \gamma_{mnp}(r, z)]$ where the γ_{mnp} are eigenfunctions of (13), but with the boundary condition $\partial\gamma_{mnp}/\partial n = 0$ on c. All properties of the α_{mnp}(orthogonality, norm, etc.) are still valid provided γ and ν are substituted for α and λ', respectively. The eigenvalues can be obtained from (15), but the admissible functions are now required to have zero normal derivative on (c).

B. Solenoidal Eigenvectors

1) Modes of Revolution: Two categories of modes will be recognized here.

a) Circular modes $\delta_{onp}\bar{u}_\phi$: There is a double infinity of these modes, corresponding to the eigenfunctions of (20) with accompanying boundary conditions. The normalization relation is (17), with β replaced by δ.

b) Meridian modes $\bar{d}_{onp}$: These eigenvectors are actually the curl of the circular electric eigenvectors. In mathematical form,

$$\bar{d}_{onp} = \text{curl}\,[\beta_{onp}\bar{u}_\phi] = \frac{1}{r}[\text{grad}\,(r\beta_{onp}) \times \bar{u}_\phi]. \tag{39}$$

It is a simple matter to check that curl $\bar{d}_{onp} = \lambda''_{onp}\beta_{onp}\bar{u}_\phi$. In consequence, curl $\bar{d}_{onp}$ vanishes on the surface of the torus, and the boundary condition $\bar{u}_n \times \text{curl}\, \bar{d}_{onp} = 0$ is satisfied there, as it should be. The normalization integral connecting $\bar{d}$ to β is similar to (22) with $\bar{c}$ and δ replaced by $\bar{d}$ and β.

2) Azimuth-Dependent Modes: The magnetic vectors are the curl of the electric vectors $\bar{e}_{mnp}$. More precisely, with $\bar{e}_{mnp}$ given by (27), $\bar{h}_{mnp}$ will be

$$\bar{h}_{mnp} = \underbrace{\cos m\phi \left[\text{curl}\left(\bar{u}_\phi \frac{r}{m}\text{div}_M \bar{c}_{mnp}\right) + \frac{m}{r}(\bar{u}_\phi \times c_{mnp})\right]}_{\text{meridian part}} + \underbrace{\sin m\phi\, \text{curl}\, \bar{c}_{mnp}}_{\text{circular part}}. \tag{40}$$

The norm of $\bar{h}$ can be evaluated with the help of (31), as:

$$\iiint_V \bar{h}_{mnp}\cdot\bar{h}_{mnp}dV = \lambda''_{mnp}\iiint \bar{e}_{mnp}\cdot\bar{e}_{mnp}dV$$
$$= \lambda''_{mnp}\pi\iint_D\left[\bar{c}_{mnp}\cdot\bar{c}_{mnp}+\frac{r^2}{m^2}(\text{div}_M\,\bar{c}_{mnp})^2\right]rdrdz.$$

If $\bar{e}_{mnp}$ has been previously normalized, the normalized magnetic eigenvector is curl $\bar{e}_{mnp}/(\lambda''_{mnp})^{1/2}$. It is sometimes desirable to calculate $\bar{h}_{mnp}$ directly without relying on a previous knowledge of $\bar{e}_{mnp}$. The relevant steps are as follows:

1) $\bar{h}_{mnp}$ will be of the form

$$\bar{h}_{mnp} = \bar{d}_{mnp}\sin m\phi + \frac{r}{m}\cos m\phi\bar{u}_\phi\,\text{div}_M\,\bar{d}_{mnp}. \quad (41)$$

2) If we go through the same motions as with the electric eigenvectors, we discover that the meridian part $\bar{d}_{mnp}$ is an eigenvector of

$$\mathcal{L}'\bar{d} = \nabla_M^2\bar{d}_{mnp} - \frac{m^2}{r^2}\bar{d}_{mnp} + \frac{2\bar{u}_r}{r}\text{div}_M\,\bar{d}_{mnp} + \lambda''_{mnp}\bar{d}_{mnp} = 0$$
$$\text{with}\begin{cases}\bar{u}_n\cdot\bar{d} = 0\\ \text{curl}_M\,\bar{d} = 0\end{cases}\text{on } (C). \quad (42)$$

Transformation $\mathcal{L}'$ is again self-adjoint and negative-definite with respect to scalar product (29). Its eigenvectors are orthogonal, and its eigenvalues are the stationary values of (30). The admissible vectors, however, must now satisfy the boundary conditions evidenced in (42). These can be written more explicitly as

$$\begin{cases}\bar{u}_n\cdot\bar{d} = d_r\cos\epsilon + d_z\sin\epsilon = 0\\ |\text{curl}_M\,\bar{d}| = \dfrac{\partial d_r}{\partial z} - \dfrac{\partial d_z}{\partial r} = 0.\end{cases}\text{on } (C), \quad (43)$$

V. Expansion in Magnetic Eigenvectors

This expansion is particularly suitable when vector $\bar{a}$ is tangent to the boundary surface. The expansion is similar to (35), with γ and $\bar{d}$ replacing α and $\bar{c}$. The only difference occurs in the expansion for v_0, which is now

$$v_0(r,z) = \frac{A_0}{r} + \sum_n\sum_p c_{onp}\,\delta_{onp}(r,z).$$

The value of the coefficients can be calculated from (1) and (35). Results are, for the sourceless vector,

$$A_0 = \frac{\iint_D v_0 drdz}{\iint_D \frac{drdz}{r}}. \quad (44)$$

$v_0(r,z)$ is $1/(2\pi r)$ times the circulation of $\bar{a}$ around the "parallel" circle through r, z. For a vector which is irrotational in the toroidal region, the circulation is constant and equal to $2\pi A_0$.

For the irrotational vectors, the coefficients are

$$A_{onp} = \frac{\int_c \gamma_{onp}(\bar{u}_n\cdot\bar{p}_0)rdc - \iint_D \gamma_{onp}\,\text{div}_M\,\bar{p}_0 rdrdz}{\nu_{onp}\iint_D \gamma_{onp}^2 rdrdz},$$
$$A_{mnp} = \frac{\int_c \gamma_{mnp}(\bar{u}_n\cdot\bar{p}_m)rdc - \iint_D \gamma_{mnp}\left(\text{div}_M\,\bar{p}_m - \frac{m}{r}v_m\right)rdrdz}{\nu_{mnp}\iint_D \gamma_{mnp}^2 rdrdz}. \quad (45)$$

A similar expression can be obtained for B_{mnp} by substituting $\bar{q}_m$ and w_m for $\bar{p}_m$ and v_m, respectively. Formulas (2) and (45) indicate that A_{onp}, A_{mnp}, and B_{mnp} vanish when $\bar{a}$ is solenoidal and perpendicular to the boundary.

For the solenoidal vectors,

$$C_{onp} = \frac{\iint_D v_0\delta_{onp} rdrdz}{\iint_D \delta_{onp}^2 rdrdz} = \frac{\iint_D \text{curl}\,(\delta_{onp}\bar{u}_\phi)\cdot\text{curl}\,(v_0\bar{u}_\phi)rdrdz}{\lambda'''_{onp}\iint_D \delta_{onp}^2 rdrdz},$$

$$D_{onp} = \frac{\iint_D \bar{p}_0\cdot\bar{d}_{onp}rdrdz}{\iint_D \bar{d}_{onp}\cdot\bar{d}_{onp}rdrdz} = \frac{\iint_D \text{curl}_M\,\bar{p}_0\cdot\text{curl}_M\,\bar{d}_{onp}rdrdz}{\lambda''_{onp}\iint_D \bar{d}_{onp}\cdot\bar{d}_{onp}rdrdz}$$
$$= \frac{\iint_D \beta_{onp}(\text{curl}\,\bar{p}_o\cdot\bar{u}_\phi)rdrdz}{\iint_D \beta_{onp}^2 rdrdz},$$

$$E_{mnp} = \frac{\iint_D \left[\bar{p}_m \cdot \bar{d}_{mnp} + \frac{r}{m} v_m \operatorname{div}_M \bar{d}_{mnp} \right] r dr dz}{\iint_D \left[\bar{d}_{mnp} \cdot \bar{d}_{mnp} + \frac{r^2}{m^2} (\operatorname{div}_M \bar{d}_{mnp})^2 \right] r dr dz}. \quad (46)$$

The numerator can be rewritten as

$$\frac{1}{\lambda''_{mnp}} \iint_D \Big\{ \operatorname{curl}_M \bar{p}_m \cdot \operatorname{curl}_M \bar{d}_{mnp}$$

$$+ \left[\operatorname{curl}_M (v_m \bar{u}_\phi) + \frac{m}{r} (\bar{u}_\phi \times \bar{p}_m \right]$$

$$\cdot \left[\operatorname{curl}_M \left(\frac{r}{m} \bar{u}_\phi \operatorname{div}_M \bar{d}_{mnp} \right) + \frac{m}{r} (\bar{u}_\phi \times \bar{d}_{mnp}) \right] \Big\} r dr dz.$$

A similar expression can be obtained for F_{mnp} by substituting $\bar{q}_m$ and w_m for $\bar{p}_m$ and v_m, respectively. It will be noticed that, according to (3), the C, D, E, and F vanish when $\bar{a}$ is irrotational.

VI. Regions Containing the Axis

In regions of the type depicted in Fig. 1(b) and 1(c), which contain parts of the axis of revolution, the Fourier expansion coefficients of a continuous function $A(r, \bar{v}, \phi)$ behave in an interesting way in the vicinity of the axis. Let the expansion be written as

$$A(r, z, \phi) = A_0(r, z) + \sum_{m=1} \sin m\phi A_m(r, z) + \sum_{m=1} \cos m\phi B_m(r, z). \quad (47)$$

If A is continuous at all points, including those situated on the axis, the limit of A as r approaches zero must be independent of ϕ. This clearly requires A_m and B_m to vanish on the axis, while the value of A reduces to $A_0(o, z)$ thereon.

Consider now a vector $\bar{a}$, continuous at all points, including those situated on the axis, and possessing a Fourier expansion of the type given in (1). By a series of simple calculations, the details of which are left to the reader, it is possible to establish the following properties of the Fourier coefficients:

1) $\bar{p}_0$ is directed along the axis;
2) v_0 vanishes on the axis;
3) $\bar{p}_1$ and $\bar{q}_1$ are purely radial on the axis, and the equalities $p_{1r} = v_1$, $q_{1r} = w_1$ hold there;
4) The coefficients $\bar{p}_m$, $\bar{q}_m$, v_m, and w_m vanish on the axis when m is larger than one.

These simple rules for scalar and vector functions allow one to foresee the behavior of functions possessing higher orders of continuity. The scalar and vector eigenelements of a cavity have continuous Laplacian and "curl curl" on the axis. Their behavior is governed by the following rules which are of great importance for practical computations:

In a simply-bounded, simply-connected cavity of the type shown in Fig. 1(b),

1) Electric and magnetic irrotational eigenvectors:
   a) When of revolution, satisfy unchanged boundary conditions on the outer contour, but the additional condition $\partial\alpha/\partial r = \partial\gamma/\partial r = 0$ on the axis.
   b) When azimuth-dependent [as described in (13)], satisfy the additional condition $\alpha = \gamma = 0$ on the axis.

2) Electric and magnetic solenoidal eigenvectors:
   a) When of the "circular mode of revolution" type [as described in (16) and (20)] satisfy the additional condition $\beta = \gamma = 0$ on the axis.
   b) When of the "meridian mode of revolution" type [as described in (18) and (39)] satisfy the additional condition $c_r = \partial c_z/\partial r = 0$ on the axis.
   c) When azimuth-dependent [as described in (26) and (42)], satisfy the additional conditions,

$$c_z = \frac{\partial c_r}{\partial r} = 0 \qquad \text{for } m = 1,$$

$$c_z = c_r = 0 \qquad \text{for } m > 1 \text{ on the axis.} \quad (48)$$

These various relations can be checked on the normal modes of the circular cylinder, which can be written down explicitly by separation of variables.[6] (See Fig 3.) The irrotational electric eigenvectors, for example, derive from scalar functions

$$\alpha_{mnp} = \sin \frac{n\pi z}{L} J_m\left(\mu_{mp} \frac{r}{a}\right)$$

where the μ_{mp} are roots of $J_m(x) = 0$. The power expansion of Bessel's function,

$$J_m(\lambda r) = \frac{(\lambda r)^m}{2^m m!}\left[1 - \frac{(\lambda r)^2}{4(m+1)} + \cdots\right] (m \text{ integer}),$$

confirms that $\partial\alpha_{onp}/\partial r = \alpha_{mnp} = 0$ on the axis. Another check is afforded by the expression for the solenoidal ϕ-dependent electric eigenvectors:

$$\bar{c}_{mnp} = -\frac{n\pi}{L} \cdot \frac{a^2}{\mu_{mp}^2} \cdot \sin \frac{n\pi z}{L} \cdot \frac{dJ_m\left(\mu_{mp} \frac{r}{a}\right)}{dr} \bar{u}_r + \cos \frac{n\pi z}{L} J_m\left(\mu_{mp} \frac{r}{a}\right) \bar{u}_z.$$

[6] See, e.g., C. G. Montgomery "Techniques of Microwave Measurements," McGraw-Hill Book Co., Inc., New York, N. Y., p. 297; 1947.

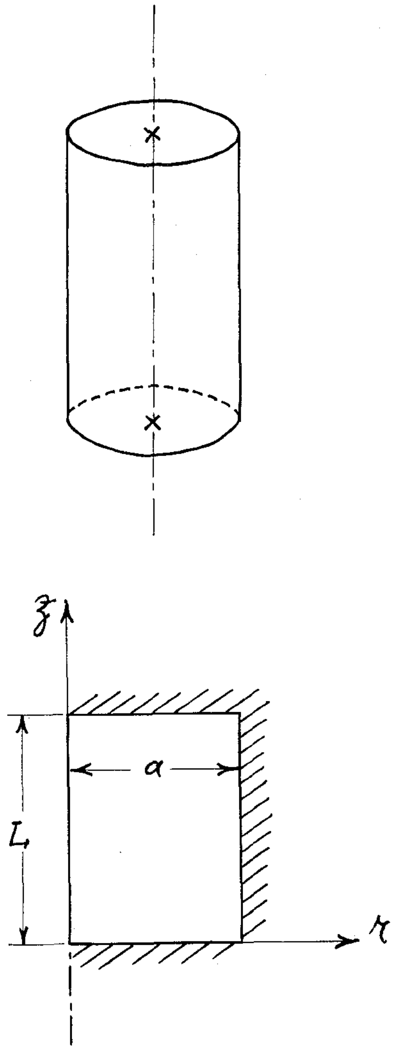

Fig. 3—Circular cylindrical cavity.

The (r, z) dependent part of the circular component is

$$v_m(r, z) = \frac{r}{m} \operatorname{div}_M \bar{c}_{mnp}$$

$$= -\frac{n\pi}{L} \cdot \frac{m}{r} \cdot \frac{a^2}{\mu_{mp}^2} \cdot \sin \frac{n\pi z}{L} \cdot J_m\left(\mu_{mp} \frac{r}{a}\right).$$

It is immediately apparent that the relevant conditions (48) are satisfied.

A last remark is in order concerning doubly-bounded volumes of the kind represented in Fig. 1(c). The electric eigenvectors considered up to now do not form a complete set unless we add the electrostatic field grad α_0 to them. This field is obtained by establishing a potential difference between the two boundary surfaces, assumed to be metallized. More precisely, α_0 is the solution of

$$\nabla_M^2 \alpha_0 = 0,$$

$$\alpha_0 = 1 \text{ on } S_1, \qquad \alpha_0 = 0 \text{ on } S_2 \qquad (49)$$

(or any multiple thereof).

Appendix

Properties of the Operator

$$\mathcal{L} = \nabla_M^2 - \frac{m^2}{r^2} + \frac{2}{r} \bar{u}_r \operatorname{div}_M$$

Scalar product (29), and the metric derived from it, define a Hilbert space. The main properties of transformation $\mathcal{L}$ are obtained from a consideration of $\langle \bar{v}, \mathcal{L}\bar{v} \rangle$, where $\bar{v}$ belongs to the domain of vectors satisfying the boundary conditions appearing in (29). If we apply (31) to $\bar{v}$, we discover that the right-hand member vanishes, so that

$$\langle \bar{v}, \mathcal{L}\bar{v} \rangle = \iint_D \left[\bar{v} \cdot \mathcal{L}\bar{v} + \frac{r^2}{m^2} \operatorname{div}_M \bar{v} \cdot \operatorname{div}_M \mathcal{L}\bar{v} \right] r\,dr\,dz$$

$$= -\iint_D \left\{ \left| \operatorname{curl}_M \bar{v} \right|^2 + \left| \frac{m}{r} \bar{u}_\phi \times \bar{v} + \operatorname{curl}_M \left(\frac{r}{m} \bar{u}_\phi \operatorname{div}_M \bar{c} \right) \right|^2 \right\} r\,dr\,dz. \qquad (50)$$

Clearly, $\langle \bar{v}, \mathcal{L}\bar{v} \rangle$ is never positive. We now want to prove that $\mathcal{L}\bar{v}_0 = 0$ implies $\bar{v}_0 = 0$, which would then make transformation (28) negative-definite. We first need to establish Helmholtz' theorem in the meridian plane. More explicitly, we want to examine the splitting of a meridian vector $\bar{P}$ into

$$\bar{P} = \text{grad } A + \bar{t} \qquad (51)$$

where grad A, the longitudinal term, is required to be perpendicular to (c), and to have the same divergence as $\bar{P}$. In other words, A must satisfy

$$\nabla_M^2 A = \operatorname{div}_M \bar{P} \qquad A = 0 \text{ on } (C).$$

It is a simple matter, with the help of Green's theorem, to show that this problem has a unique solution, and that the longitudinal term vanishes when $\operatorname{div}_M \bar{P} = 0$. It is also a simple matter, using Stokes' theorem in the meridian plane, to show that each meridian vector for which curl $\bar{P} = 0$ can be put in the form $\text{grad}_M\, \theta$. If, in addition, $\bar{P}$ is perpendicular to the boundary c, potential θ is nothing but the function A appearing in (51). The sources of $\bar{t}$ are, consequently, the curl of $\bar{P}$ and the tangential components of $\bar{P}$. When $\mathcal{L}\bar{v}_0 = 0$, the left-hand member of (51) vanishes; this implies that the squares in the second member also vanish, and, in particular, that curl $\bar{v}_0 = 0$. Letting $\bar{v}_0 = \text{grad } A$, it is found that A must satisfy

$$\mathcal{L}\bar{v}_0 = \nabla_M^2 \text{ grad } A - \frac{m^2}{r^2} \text{ grad } A + \frac{2}{r} \bar{u}_r \nabla_M^2 A = 0$$

with

$$A = \nabla_M^2 A = 0 \text{ on } (C).$$

Projection of $\mathcal{L}\bar{v}_0$ on the z axis indicates that

$$\frac{\partial}{\partial z}\left[\nabla_M^2 A - \frac{m^2}{r^2} A\right] = 0.$$

In consequence, $\nabla_M^2 A - (m^2/r^2)A$ has a constant value along a parallel to the z axis. This value must be zero, because

$$\nabla_M^2 A - \frac{m^2}{r^2} A = 0 \qquad (52)$$

on c. It follows that (52) is valid over the whole area D. An application of Green's theorem shows that

$$\iint_D \left\{\frac{m^2}{r^2} A^2 + |\text{grad } A|^2\right\} r\,dr\,dz = 0$$

so that both A and $\bar{v}_0$ must vanish.

The self-adjoint character of $\mathcal{L}$ (*i.e.*, $\langle \bar{c}, \mathcal{L}\bar{d}\rangle = \langle \bar{d}, \mathcal{L}\bar{c}\rangle$) can be quickly established by using a relation derived from the three-dimensional Green's theorem (32):

$$\iint_D \left\{\bar{Q}\cdot\left(\nabla_M^2 \bar{P} - \frac{m^2}{r^2}\bar{P} + \frac{2mg}{r^2}\bar{u}_r\right) + h\left(\nabla_M^2 g - \frac{m^2+1}{r^2} g + \frac{2mc_r}{r^2}\right) + \left(\text{div}_M \bar{P} - \frac{mg}{r}\right)\cdot\left(\text{div}_M \bar{Q} - \frac{mh}{r}\right) + \text{curl}_M \bar{P}\cdot\text{curl}_M \bar{Q} + \left[\text{curl}_M (g\bar{u}_\phi) + \frac{m}{r}(\bar{u}_\phi \times \bar{P})\right]\cdot\left[\text{curl}_M (h\bar{u}_\phi) + \frac{m}{r}(\bar{u}_\phi \times \bar{Q})\right]\right\} r\,dr\,dz$$
$$= \int_c \left\{(\bar{u}_n \times \bar{Q})\cdot\text{curl}_M \bar{P} - \frac{hm}{r}(\bar{u}_n\cdot\bar{P}) - h\bar{u}_t\cdot\text{curl}_M (g\bar{u}_\phi) + (\bar{u}_n\cdot\bar{Q})\left(\text{div}_M \bar{P} - \frac{mg}{r}\right)\right\} r\,dc. \qquad (53)$$

If we use this relation twice, setting $g = r/m \ \text{div}_M \bar{P}$, $h = r/m \ \text{div}_M \bar{Q}$ and subtracting, we obtain, since

$$\text{div}_M\left[\nabla_M^2 \bar{c} - \frac{m^2\bar{c}}{r^2} + \frac{2}{r}\cdot\bar{u}_r \,\text{div}_M \bar{c}\right]$$
$$= \frac{m}{r}\nabla_M^2\left(\frac{r\,\text{div}_M \bar{c}}{m}\right) + \frac{2m^2}{r^3} c_r - \frac{m^2+1}{r^2}\text{div}_M \bar{c}$$
$$= \nabla_M^2 (\text{div}_M \bar{c}) - \frac{m^2}{r^2}\text{div}_M \bar{c} + \frac{2m^2}{r^3} c_r + \frac{2}{r}\frac{\partial}{\partial r}(\text{div}_M \bar{c}), \qquad (54)$$

the relation,

$$\iint_D \left[\bar{Q}\cdot\mathcal{L}\bar{P} - \bar{P}\cdot\mathcal{L}\bar{Q} + \frac{r^2}{m^2}\text{div}_M \bar{Q}\,\text{div}_M \mathcal{L}\bar{P} - \frac{r^2}{m^2}\text{div}_M \bar{P}\,\text{div}_M \mathcal{L}\bar{Q}\right] r\,dr\,dz = 0. \qquad (55)$$

The second member in (53) vanishes because of the boundary conditions. Eq. (55) is nothing but $\langle \bar{c}, \mathcal{L}\bar{d}\rangle - \langle \bar{d}, \mathcal{L}\bar{c}\rangle = 0$, the relation we set out to prove.

A Printed Circuit Balun for Use with Spiral Antennas*

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Summary—**A novel printed circuit balun is described which is particularly well suited to applications where space is at a premium. The design utilizes unshielded strip transmission line, but is readily adaptable to all of the common printed circuit transmission line techniques. When the balun is housed within the cavity of a spiral antenna, boresight error is virtually eliminated, ellipticity ratios of less than 2 db are maintained over an azimuth angle greater than ±60°, and the input standing-wave ratio is less than 2:1 over an octave frequency range. Experimental results are given and additional applications are described.**

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I. Introduction

A BALUN is a term used by antenna engineers to describe a device which transforms an unbalanced to a balanced transmission line. To the microwave engineer, the same device might be called a ratrace, magic tee, or more generally a hybrid. In lumped circuit applications, we also find a similar device used in conjunction with balanced mixers, phase detectors, and single-sideband modulators, to name a few. However, regardless of what the device is called, the operation will appear to be basically similar provided the analysis is made using a compatible frame of reference. One particularly powerful tool used at microwaves